

# On the $B_c$ leptonic decay constant

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## Abstract

We give a review and present a comprehensive calculations for the leptonic constant  $f_{B_c}$  of the low-lying pseudoscalar and vector states of  $B_c$ -meson in the framework of static and QCD-motivated nonrelativistic potential models taking into account the one-loop and two-loop QCD corrections in the short distance coefficient that governs the leptonic constant of  $B_c$  quarkonium system. Further, we use the scaling relation to predict the leptonic constant of the nS-states of the  $\bar{b}c$  system. Our results are compared with other models to gauge the reliability of the predictions and point out differences.

## I. INTRODUCTION

The  $B_c$  mesons provide a unique window into the heavy quark dynamics. Although they are intermediate to the charmonium and bottomonium systems the properties of  $B_c$  mesons are a special case in quarkonium spectroscopy as they are the only quarkonia consisting of heavy quarks with different flavours. Because they carry flavour they cannot annihilate into gluons so are more stable and excited  $B_c$ -states lying below BD (and BD\* or B\*D) threshold can only undergo radiative or hadronic transitions to the ground state  $B_c$  which then decays weakly. There are two sets of S-wave states, as many as two P-wave multiplets (the 1P and some or all of the 2P) and one D-wave multiplet lying below BD threshold for emission of B and D mesons. As well, the F-wave multiplet is sufficiently close to threshold that they may also be relatively narrow due to angular momentum barrier suppression of the Zweig allowed strong decays. However, the spectrum and properties of these states have been calculated various times in the framework of heavy quarkonium theory [1].

The discovery of the  $B_c$  meson by the Collider Detector at Fermilab (CDF) Collaboration [2] in the channel  $B_c \rightarrow J/\psi l\nu$  ( $l = e, \mu, \tau$ ) with low-lying level of pseudoscalar mass  $M_{B_c} = 6.4^{+0.39}_{-0.13}$  GeV and lifetime  $\tau_{B_c} = 0.46^{+0.18}_{-0.16} \pm 0.03$  ps confirmed the theoretical predictions regarding various  $B_c$  meson properties, spectroscopy, production and decay channels [1].

The description of various aspects of  $B_c$  meson physics [3] has received recently a lot of attention both from theoretical as well as experimental sides. It could offer a unique probe to check the perturbative QCD predictions more precisely and lead to a new information about the confinement scale inside hadrons. The higher  $\bar{b}c$ -meson state cascades down through lower energy  $\bar{b}c$  state via hadronic or electromagnetic transitions to the pseudoscalar ground state  $B_c$  meson. The theoretical study of the pure leptonic decays of the  $B_c$  meson such as  $B_c \rightarrow l\nu_l$  can be used to determine the leptonic decay constant  $f_{B_c}$  [4], as well as the fundamental parameters in the standard Model (SM). Hence, it is the only meson within SM which is composed of two nonrelativistic heavy quarks of different flavors: open charm and bottom quarks. Its spectroscopy production mechanisms and decays differ significantly from those of charmonium  $J/\psi$  and upsilon  $\Upsilon$  as well as hadrons with one heavy quark. Indeed, this meson is also a long lived system decaying through electroweak interactions. It stands among the families of  $\bar{c}c$  and  $\bar{b}b$  and thus could be used to study both quantitatively and conceptually existing effective low energy frameworks for the description of bound state heavy quark dynamics, like NRQCD [5], pNRQCD [6] and vNRQCD [7].

The calculation of the  $B_c$  leptonic decay constant  $f_{B_c}$  can be carried out either using QCD-based methods, such as lattice QCD [8], QCD sum rules [9], or adopting some constituent quark models [1]. So far, lattice QCD has only been employed to calculate the  $B_c$  purely leptonic width. As the QCD sum rules, the  $B_c$  leptonic constant, as well as the matrix elements relevant for the semileptonic decays were computed. Moreover, the leptonic constant was also been calculated by using the nonrelativistic (NR) potential model [1].

Ikhdaire *et al.* [10] applied the NR form of the statistical model to calculate the spectroscopy, decay constant and some other properties of the heavy mesons, including the  $\bar{b}c$  system, using a class of three static quarkonium potentials. Davies *et al.* [8] predicted the leptonic constant of the lowest state of the  $B_c$  system with the recent lattice calculations. Eichten and Quigg [1] gave a more comprehensive account of the decays of the  $B_c$  system that was based on the QCD-motivated potential of Buchmüller and Tye [11]. Gershtein *et al.* [1] also published a detailed account of the energies and decays of the  $B_c$  system using

a QCD sum-rule calculations [9]. Fulcher [2] also calculated the leptonic constant of the  $B_c$  quarkonium system using the treatment of the spin-dependent potentials to the full radiative one-loop level and thus included effects of the running coupling constant in these potentials. Furthermore, he also used the renormalization scheme developed by Gupta and Radford [12]. Ebert *et al.* [1] have also calculated the leptonic constant of the pseudoscalar and vector  $B_c$  quarkonium system using a relativistic model and then compared their calculation with the NR model. Kiselev *et al.* [13-15] calculated the leptonic constant of the pseudoscalar meson  $B_c$  in the framework of QCD-motivated potential models taking into account the one-loop [13], the two-loop [14] correction matching and scaling relation (SR) [15]. Capstick and Godfrey [16] predicted the result of  $f_{B_c} = 410 \text{ MeV}$  for the decay constant of the  $B_c$  meson using the Mock-Meson approach or other relativistic quark models. Ikhdair and Sever [17] have calculated the decay constant in the nonrelativistic and semi-relativistic quark model using the shifted large-N expansion technique. Finally, Godfrey [18] has calculated the spectroscopy of  $B_c$  meson in the relativized quark model. Motyka and Zalewski [18] proposed a nonrelativistic Hamiltonian with plausible spin dependent corrections for the quarkonia below their respective strong decay threshold. They found the decay constants for the ground state and for the first excited S-state of  $\bar{b}c$  quarkonium to be  $f_{B_c} = 435 \text{ MeV}$  and  $f_{B_c} = 315 \text{ MeV}$ , respectively..

We have made this study mostly a fully treatment for the quarkonium potential models used in the literature [1,17] in calculating the  $B_c$  leptonic constant based on the NR potential model, (cf. e.g. [17] and references contained there), and applied with a great success to fit the entire heavy quarkonium spectroscopy [17,19]. We insist upon strict flavor-independence of its parameters. We also use the potential models to give a simultaneous account of the properties of the  $\bar{c}c$ ,  $\bar{b}b$  and  $\bar{b}c$  quarkonium systems.

The contents of this paper is as follows: In section II, we present briefly the solution of the Schrödinger equation for the  $\bar{b}c$  quarkonium mass spectrum using the shifted large-N-expansion technique (SLNET). In section III, we introduce the necessary expressions for the one-loop and two-loop corrections to the leptonic constant available to moment. Further,

in section IV we present the phenomenological and the QCD-motivated potentials used in the present work. Finally, section V contains our calculations and remarks made for the leptonic constant in our approach and scaling relation.

## II. WAVE EQUATION

In this section we consider the N-dimensional space Schrödinger equation for any spherically symmetric potential  $V(\mathbf{r})$ . If  $\psi(\mathbf{r})$  denotes the Schrödinger's wave function, a separation of variables  $\psi(\mathbf{r}) = Y_{l,m}(\theta, \phi)u(r)/r^{(N-1)/2}$  gives the following radial equation (in units  $\hbar = c = 1$ ) [17,19,20]

$$\left\{ -\frac{1}{4\mu} \frac{d^2}{dr^2} + \frac{[\bar{k} - (1-a)][\bar{k} - (3-a)]}{16\mu r^2} + V(r) \right\} u(r) = E_{n,l} u(r), \quad (1)$$

where  $\mu = (m_{\bar{q}_1} m_{q_2}) / (m_{\bar{q}_1} + m_{q_2})$  is the reduced mass for the two interacting particles. Here  $E_{n,l}$  denotes the Schrödinger binding energy and  $\bar{k} = N + 2l - a$ , with  $a$  representing a proper shift to be calculated later on and  $l$  is the angular quantum number. We follow the shifted  $1/N$  or  $1/\bar{k}$  expansion method [17] by defining

$$V(y(r_0)) = \frac{1}{Q} \sum_{m=0}^{\infty} \left( \frac{d^m V(r_0)}{dr_0^m} \right) \frac{(r_0 y)^m}{m!} \bar{k}^{-(m-4)/2}, \quad (2)$$

and also the energy eigenvalue expansion [17,19]

$$E_{n,l} = \sum_{m=0}^{\infty} \frac{\bar{k}^{(2-m)}}{Q} E_m, \quad (3)$$

where  $x = \bar{k}^{1/2}(r/r_0 - 1)$  with  $r_0$  is an arbitrary point where the Taylor expansion is being performed about and  $Q$  is a scale to be set equal to  $\bar{k}^2$  at the end of our calculations. Inserting equations (2) and (3) into equation (1) gives

$$\begin{aligned} & \left[ -\frac{1}{4\mu} \frac{d^2}{dy^2} + \frac{1}{4\mu} \left( \frac{\bar{k}}{4} - \frac{(2-a)}{2} + \frac{(1-a)(3-a)}{4\bar{k}} \right) \times \sum_{m=0}^{\infty} (-1)^m \frac{(1+m)}{\bar{k}^{m/2}} y^m \right. \\ & \left. + \frac{r_0^2}{Q} \sum_{m=0}^{\infty} \left( \frac{d^m V(r_0)}{dr_0^m} \right) \frac{(r_0 y)^m}{m!} \bar{k}^{-(m-2)/2} \right] \chi_{n_r}(y) = \xi_{n_r} \chi_{n_r}(y), \end{aligned} \quad (4)$$

where the final analytic expression for the  $1/\bar{k}$  expansion of the energy eigenvalues appropriate to the Schrödinger particle is [17]

$$\xi_{n_r} = \frac{r_0^2}{Q} \sum_{m=0}^{\infty} \bar{k}^{(1-m)} E_m, \quad (5)$$

where  $n_r$  is the radial quantum number. Hence, we formulate the SLNET (expansion as  $1/\bar{k}$ ) for the nonrelativistic motion of spinless particle bound in spherically symmetric potential  $V(r)$ . The resulting eigenvalue of the N-dimensional Schrödinger equation is written as [17]

$$\begin{aligned} \xi_{n_r} = & \bar{k} \left[ \frac{1}{16\mu} + \frac{r_0^2 V(r_0)}{Q} \right] + \left[ (n_r + \frac{1}{2}) \omega - \frac{(2-a)}{8\mu} \right] \\ & + \frac{1}{\bar{k}} \left[ \frac{(1-a)(3-a)}{16\mu} + \alpha^{(1)} \right] + \frac{\alpha^{(2)}}{\bar{k}^2}, \end{aligned} \quad (6)$$

where  $\alpha^{(1)}$  and  $\alpha^{(2)}$  are two useful expressions given by Imbo et al [20]. Comparing equation (5) with (6) yields

$$E_0 = V(r_0) + \frac{Q}{16\mu r_0^2}, \quad (7)$$

$$E_1 = \frac{Q}{r_0^2} \left[ \left( n_r + \frac{1}{2} \right) \omega - \frac{(2-a)}{8\mu} \right], \quad (8)$$

$$E_2 = \frac{Q}{r_0^2} \left[ \frac{(1-a)(3-a)}{16\mu} + \alpha^{(1)} \right], \quad (9)$$

and

$$E_3 = \frac{Q}{r_0^2} \alpha^{(2)}. \quad (10)$$

The quantity  $r_0$  is chosen so as to minimize the leading term,  $E_0$  [17,19], that is,

$$\frac{dE_0}{dr_0} = 0 \quad \text{and} \quad \frac{d^2E_0}{dr_0^2} > 0. \quad (11)$$

Therefore,  $r_0$  satisfies the relation

$$Q = 8\mu r_0^3 V'(r_0), \quad (12)$$

and to solve for the shifting parameter  $a$ , the next contribution to the energy eigenvalue  $E_1$  is chosen to vanish [20] so that the second- and third-order corrections are very small compared with the leading term contribution. The energy states are calculated by considering the leading term  $E_0$  and the second-order and third-order corrections, it implies the shifting parameter

$$a = 2 - (2n_r + 1) \left[ 3 + \frac{r_0 V''(r_0)}{V'(r_0)} \right]^{1/2}. \quad (13)$$

Therefore, the Schrödinger binding energy to the third order is

$$E_{n,l} = V(r_0) + \frac{r_0 V'(r_0)}{2} + \frac{1}{r_0^2} \left[ \frac{(1-a)(3-a)}{16\mu} + \alpha^{(1)} + \frac{\alpha^{(2)}}{\bar{k}} + O\left(\frac{1}{\bar{k}^2}\right) \right], \quad (14)$$

Once the problem is collapsed to its actual dimension ( $N = 3$ ), it simply rests the task of relating the coefficients of our equation to the one-dimensional anharmonic oscillator in order to read the energy spectrum. For the  $N = 3$  physical space, the equation (1) restores its three-dimensional form. Thus, with the choice  $\bar{k} = \sqrt{Q}$  which rescales the potential, we derive an analytic expression that satisfies  $r_0$  in equations (12)-(13) as

$$1 + 2l + (2n_r + 1) \left[ 3 + \frac{r_0 V''(r_0)}{V'(r_0)} \right]^{1/2} = \left[ 8\mu r_0^3 V'(r_0) \right]^{1/2}. \quad (15)$$

where  $n_r = 0, 1, 2, \dots$  stands for the radial quantum number and  $l = 0, 1, 2, \dots$  stands for the angular quantum number. Once  $r_0$  is being determined through equation (15), the Schrödinger binding energy of the  $\bar{q}_1 q_2$  system in equation (14) becomes relatively simple and straightforward. We finally obtain the total Schrödinger mass binding energy for spinless particles as

$$M(\bar{q}_1 q_2) = m_{\bar{q}_1} + m_{q_2} + 2E_{n,l}. \quad (16)$$

As stated before in [17,19,20], for any fixed  $n$  the computed energies become more convergent as  $l$  increases. This is expected since the expansion parameter  $1/N$  or  $1/\bar{k}$  becomes smaller as  $l$  becomes larger since the parameter  $\bar{k}$  is proportional to  $n$  and appears in the denominator in higher-order correction.

On the other hand, the spin dependent correction to the nonrelativistic Hamiltonian, which is responsible for the hyperfine splitting of the 1S-state mass level is generally used in the form (cf. e.g. Refs. [1,17,21])<sup>1</sup>

$$\Delta E_{\text{HF}} = \frac{32\pi\alpha_s(\mu)}{9m_c m_b} |\psi_{1S}(0)|^2. \quad (17)$$

Like most authors (cf. e.g. [1,17,21]), we determine the coupling constant  $\alpha_s(m_c)$  from the well measured experimental hyperfine splitting of the 1S( $c\bar{c}$ ) state value

$$\Delta E_{\text{HF}} = M_{J/\psi} - M_{\eta_c} = 117 \pm 2 \text{ MeV}. \quad (18)$$

We use the value of the coupling constant to reproduce the spin-averaged data (SAD) or center-of-gravity (c.o.g.) of the lowest charmonium state value, that is,  $\overline{M}_\psi(1S)$ . In order to apply this formula one needs the value of the wave function at the origin, this is obtained by solving the Schrödinger equation with the nonrelativistic Hamiltonian and the coupling constant  $\alpha_s(\mu)$ . The radial wave function at the origin [21] is determined by

$$|R_{1S}(0)|^2 = 2\mu \left\langle \frac{dV(r)}{dr} \right\rangle. \quad (19)$$

where  $|R_{1S}(0)|^2 = 4\pi |\psi_{1S}(0)|^2$ . Hence, the total mass of the low-lying pseudoscalar  $B_c$ -state is given by the expression

$$M_{B_c}(0^-) = m_c + m_b + 2E_{1,0} - 3\Delta E_{\text{HF}}/4, \quad (20)$$

and also for the vector  $B_c^*$ -state

$$M_{B_c^*}(1^-) = m_c + m_b + 2E_{1,0} + \Delta E_{\text{HF}}/4. \quad (21)$$

Finally, the square-mass difference can be simply found by using the expression

$$\Delta M^2 = M_{B_c^*}(1^-)^2 - M_{B_c}(0^-)^2 = 2\Delta E_{\text{HF}} [m_c + m_b + 2E_{1,0} - \Delta E_{\text{HF}}/4]. \quad (22)$$

<sup>1</sup>To the moment, the only measured splitting of  $nS$ -levels is that of  $\eta_c$  and  $J/\psi$ , which allows us to evaluate the so-called SAD using  $\overline{M}_\psi(1S) = (3M_{J/\psi} + M_{\eta_c})/4$  and also  $\overline{M}(nS) = M_V(nS) - (M_{J/\psi} - M_{\eta_c})/4n$  [13,21].

### III. LEPTONIC DECAY CONSTANT OF THE $B_C$ -MESON

The study of the heavy quarkonium system has played a vital role in the development of the QCD. Some of the earliest applications of perturbative QCD were calculations of the decay rates of charmonium [22]. These calculations were based on the assumption that, in the NR limit, the decay rate factors into a short-distance (SD) perturbative part associated with the annihilation of the heavy quark and antiquark and a long-distance (LD) part associated with the quarkonium wavefunction. Calculations of the annihilation decay rates of heavy quarkonium have recently been placed on a solid theoretical foundation by Bodwin *et al.* [23]. Their approach is based on NRQCD, and effective field theory that is equivalent to QCD to any given order in the relative velocity  $v$  of the heavy quark and antiquark [24]. Using NRQCD [5] to separate the SD and LD effects, Bodwin *et al.* derived a general factorization formula for the inclusive annihilation decay rates of heavy quarkonium. The SD factors in the factorization formula can be calculated using pQCD, and the LD factors are defined rigorously in terms of the matrix elements of NRQCD that can be estimated using lattice calculations. It applies equally well to S-wave, P-wave, and higher orbital-angular-momentum states, and it can be used to incorporate relativistic corrections to the decay rates.

In the NRQCD [25] approximation for the heavy quarks, the calculation of the leptonic decay constant for the heavy quarkonium with the two-loop accuracy requires the matching of NRQCD currents with corresponding full-QCD axial-vector currents [14]

$$\mathcal{J}^\lambda \Big|_{\text{NRQCD}} = -\chi_b^\dagger \psi_c v^\lambda \quad \text{and} \quad J^\lambda \Big|_{\text{QCD}} = \bar{b} \gamma^\lambda \gamma_5 c, \quad (23)$$

where  $b$  and  $c$  are the relativistic bottom and charm fields, respectively,  $\chi_b^\dagger$  and  $\psi_c$  are the NR spinors of anti-bottom and charm and  $v^\lambda$  is the four-velocity of heavy quarkonium. The NRQCD [25] lagrangian describing the  $B_c$ -meson bound state dynamics is

$$\mathcal{L}_{\text{NRQCD}} = \mathcal{L}_{\text{light}} + \psi_c^\dagger \left( iD_0 + \mathbf{D}^2/(2m_c) \right) \psi_c + \chi_b^\dagger \left( iD_0 - \mathbf{D}^2/(2m_b) \right) \chi_b + \dots, \quad (24)$$

where  $\mathcal{L}_{\text{light}}$  is the relativistic lagrangian for gluons and light quarks. The two-component spinor field  $\psi_c$  annihilates charm quarks, while  $\chi_b$  creates bottom anti-quarks. The relative velocity  $v$  of heavy quarks inside the  $B_c$ -meson provides a small parameter that can be used as a nonperturbative expansion parameter. To express the decay constant  $f_{B_c}$  in terms of NRQCD matrix elements we express  $J^\lambda|_{\text{QCD}}$  in terms of NRQCD fields  $\psi_c$  and  $\chi_b$ . The  $\lambda = 0$  current-component contributes to the matrix element and consequently the  $J^\lambda|_{\text{QCD}}$  has the following operator expansion

$$\langle 0 | \bar{b} \gamma^\lambda \gamma_5 c | B_c(\mathbf{P}) \rangle = i f_{B_c} P^\lambda, \quad (25)$$

where  $|B_c(\mathbf{P})\rangle$  is the state of the  $B_c$ -meson with four-momentum  $P$ . Only the  $\lambda = 0$  component contributes to the matrix element (25) in the rest frame of the  $B_c$ -meson. It has the standard covariant normalization

$$\frac{1}{(2\pi)^3} \int \psi_{B_c}^*(p') \psi_{B_c}(p) d^3 p = 2E \delta^{(3)}(p' - p), \quad (26)$$

and its phase has been chosen so that  $f_{B_c}$  is real and positive. In NRQCD, the matching yields

$$\bar{b} \gamma^0 \gamma_5 c = K_0 \chi_b^\dagger \psi_c + K_2 (\mathbf{D} \chi_b)^\dagger \cdot \mathbf{D} \psi_c + \dots, \quad (27)$$

where  $K_0 = K_0(m_c, m_b)$  and  $K_2 = K_2(m_c, m_b)$  are Wilson SD coefficients. They can be determined by matching perturbative calculations of the matrix element  $\langle 0 | \bar{b} \gamma^0 \gamma_5 c | B_c \rangle$  that is mainly resulting from the operator  $\chi_b^\dagger \psi_c$  in

$$\langle 0 | \bar{b} \gamma^0 \gamma_5 c | B_c \rangle|_{\text{QCD}} = K_0 \langle 0 | \chi_b^\dagger \psi_c | B_c \rangle|_{\text{NRQCD}} + K_2 \langle 0 | (\mathbf{D} \chi_b)^\dagger \cdot \mathbf{D} \psi_c | B_c \rangle|_{\text{NRQCD}} + \dots, \quad (28)$$

where the matrix element on the left side of (28) is taken between the vacuum and the state  $|B_c\rangle$ . Hence, equation (28) can be estimated as

$$|\langle 0 | \chi_b^\dagger \psi_c | B_c \rangle|^2 \simeq \frac{3M_{B_c}}{\pi} |R_{1S}(0)|^2. \quad (29)$$

Onishchenko and Veretin [25] calculated the matrix elements on both sides of equation (28) up to  $\alpha_s^2$  order. In one-loop calculation, the SD-coefficients are

$$K_0 = 1 \quad \text{and} \quad K_2 = -\frac{1}{8\mu^2}, \quad (30)$$

with  $\mu$  defined after equation (1). Further, Braaten and Fleming in their work [26] calculated the perturbation correction to  $K_0$  up to order-  $\alpha_s$  (one-loop correction) as

$$K_0 = 1 + c_1 \frac{\alpha_s(\mu)}{\pi}, \quad (31)$$

with  $c_1$  being calculated in Ref. [26] as

$$c_1 = - \left[ 2 - \frac{m_b - m_c}{m_b + m_c} \ln \frac{m_b}{m_c} \right]. \quad (32)$$

Finally, the leptonic decay constant for the one-loop calculations is

$$f_{B_c}^{(1\text{-loop})} = \left[ 1 - \frac{\alpha_s(\mu)}{\pi} \left[ 2 - \frac{m_b - m_c}{m_b + m_c} \ln \frac{m_b}{m_c} \right] \right] f_{B_c}^{\text{NR}}, \quad (33)$$

where the NR leptonic constant [27] is

$$f_{B_c}^{\text{NR}} = \sqrt{\frac{3}{\pi M_{B_c}}} |R_{1S}(0)| \quad (34)$$

and  $\mu$  is any scale of order  $m_b$  or  $m_c$  of the running coupling constant. On the other hand, the calculations of two-loop correction in the case of vector current and equal quark masses was done in [28]. Further, Onishchenko and Veretin [25] extended the work of Ref. [28] into the non-equal mass case. They found an expression for the two-loop QCD corrections to  $B_c$ -meson leptonic constant given by

$$K_0(\alpha_s, M/\mu) = 1 + c_1(M/\mu) \frac{\alpha_s(M)}{\pi} + c_2(M/\mu) \left( \frac{\alpha_s(M)}{\pi} \right)^2 + \dots, \quad (35)$$

where  $c_2(M/\mu)$  is the two-loop matching coefficient and with  $c_{1,2}$  are explicitly given in Eq. (32) and (Ref. [25]; see equations (16)-(20) therein), respectively. In the case of  $B_c$ -meson and pole quark masses ( $m_b = 4.8$  GeV,  $m_c = 1.65$  GeV), they found

$$f_{B_c}^{(2\text{-loop})} = \left[ 1 - 1.48 \left( \frac{\alpha_s(m_b)}{\pi} \right) - 24.24 \left( \frac{\alpha_s(m_b)}{\pi} \right)^2 \right] f_{B_c}^{\text{NR}}. \quad (36)$$

Therefore, the two-loop corrections are large and constitute nearly 100% of one-loop correction as stated in Ref. [25].

## IV. SOME POTENTIAL MODELS

The potential models are suitable for the phenomenological studies, because they can reproduce the model formulae or numbers for the quantities, used as input values (the level masses, for example). Hence the potential models can be considered as phenomenological meaningful fittings of some experimental values, but they can not restore a true potential, that does not exist due to the nonperturbative effects [15].

### A. Static potentials

It is easy to see that most phenomenological static potentials used in the Scrödinger equation in [17] may be gathered up in a general form [17,29]

$$V(r) = -ar^{-\alpha} + br^{\beta} + V_0 \quad 0 \leq \alpha, \beta \leq 1, \quad a, b \geq 0, \quad (37)$$

where  $V_0$  may be of either sign. Here we assume that the effective  $\bar{q}_1 q_2$  potential consists of two terms, one of which,  $V_V(r) = -ar^{-\alpha}$ , transforms like a time-component of a Lorentz 4-vector and the other,  $V_S(r) = br^{\beta}$ , like a Lorentz scalar. These static quarkonium potentials are monotone nondecreasing, and concave functions which satisfy the condition

$$V'(r) > 0 \text{ and } V''(r) \leq 0. \quad (38)$$

At least ten potentials of this general form, but with various values of the parameters, have been proposed in the literature (cf. [17] and references contained there). The generality (37) comprises the following five types of potentials used in the literature: Some of these potentials have  $\alpha = \beta$  as in the Cornell ( $\alpha = \beta = 1$ ); Song-Lin ( $\alpha = \beta = \frac{1}{2}$ ) and Turin ( $\alpha = \beta = \frac{3}{4}$ ) potentials with same sets of fitting parameters used in our previous works [17]. Further, Song [30] has also used a potential with  $\alpha = \beta = \frac{2}{3}$ .

On the other hand, potentials with  $\alpha \neq \beta$  have also been popular as

### 1. Martin potential

The phenomenological power-law potential proposed by Martin (cf. e.g. [15,17]) has  $\alpha = 0$ ,  $\beta = 0.1$  of the form

$$V_M(r) = b_M(\Lambda_M r)^{0.1} + c_M, \quad (39)$$

is labeled as Martin's potential [15] with a given set of adjustable parameters

$$[b_M, c_M, \Lambda_M] = [6.898 \text{ GeV}^{1.1}, -8.093 \text{ GeV}, 1 \text{ GeV}], \quad (40)$$

and quark masses

$$[m_c, m_b] = [1.800 \text{ GeV}, 5.174 \text{ GeV}]. \quad (41)$$

(potential units are also in GeV).

### 2. Logarithmic potential

A Martin's power-law potential turns into the logarithmic potential of Quigg and Rosner [15,17] corresponds to  $\alpha = \beta \rightarrow 0$  and it takes the general form

$$V_L(r) = b_L \ln(\Lambda_L r) + c_L, \quad (42)$$

with an adjustable set of parameters

$$[b_L, c_L, \Lambda_L] = [0.733 \text{ GeV}, -0.6631 \text{ GeV}, 1 \text{ GeV}], \quad (43)$$

and quark masses

$$[m_c, m_b] = [1.500 \text{ GeV}, 4.905 \text{ GeV}]. \quad (44)$$

The potential forms in (39), and (42) were also used by Kiselev in [15]. Further, they were also been used for  $\psi$  and  $\Upsilon$  data probing  $0.1 \text{ fm} < r < 1 \text{ fm}$  region [13].

Further, Motyka and Zalewski [29] used a nonrelativistic potential with  $\alpha = 1$  and  $\beta = \frac{1}{2}$  for the  $\bar{b}b$  quarkonium and then applied it to the  $\bar{c}c$  and  $\bar{b}c$  quarkonium systems.

Grant, Rosner, and Rynes [31] have suggested  $\alpha = 0.045$ ,  $\beta = 0$ . Heikkila, Törnquist and Ono [32] tried  $\alpha = 1$ ,  $\beta = \frac{2}{3}$ . Some very successful potentials known from the literature are not of this type. Examples are the Indiana potential [33] and the Richardson potential [34].

## B. QCD-motivated potentials

We use two types of the QCD-motivated potentials: The Igi-Ono (IO) and an improved Chen-Kuang (CK) potential models. The details of these potentials can be traced in our previous works in [17].

## V. CALCULATIONS AND CONCLUSION

Based on our previous works [17], we determine the position of the charmonium center-of-gravity (c.o.g.)  $\overline{M}_\psi(1S)$  mass spectrum and its hyperfine splittings by fixing the coupling constant  $\alpha_s(m_c)$  in (17) for each central potential. Further, we calculate the corresponding low-lying (c.o.g.)  $\overline{M}_\Upsilon(1S)$  and consequently the low-lying  $\overline{M}_{B_c}(1S)$ .<sup>2</sup> In Table I, we estimate the radial wave function of the low-lying state of the  $\bar{b}c$  system, so that

$$|R_{1S}(0)| = 1.18 - 1.24 \text{ GeV}^{3/2}, \quad (45)$$

for the set of the central potentials given in section IVA. Further, in Table II we present our results for the NR leptonic constant  $f_{B_c}^{\text{NR}} = 466 \pm 19$  MeV and  $f_{B_c^*}^{\text{NR}} = 464 \pm 19$  MeV as an estimation of the potential models without the matching. Our results are compared with those of Gershtein, Likhoded and Sabespi [35], who used Martin's potential and with those of Jones and Woloshyn [36]. Moreover, the one-loop correction  $f_{B_c}^{(1-\text{loop})}$  and the two-loop correction  $f_{B_c}^{(2-\text{loop})}$  are also given in Table III. Therefore, in the view of our results, our prediction for the one-loop calculation is

<sup>2</sup>Kiselev *et al.* [15] have taken into account that  $\Delta M_\Upsilon(1S) = \frac{\alpha_s(\Upsilon)}{\alpha_s(\psi)} \Delta M_\psi(1S)$  with  $\alpha_s(\Upsilon)/\alpha_s(\psi) \simeq 3/4$ . On the other hand, Motyka and Zalewski [29] also found  $\frac{\alpha_s(m_b^2)}{\alpha_s(m_c^2)} \simeq 11/18$ .

$$f_{B_c}^{(1\text{-loop})} = 408 \pm 16 \text{ MeV}, \quad (46)$$

and also for two-loop calculation

$$f_{B_c}^{(2\text{-loop})} = 315 \pm 50 \text{ MeV}. \quad (47)$$

Our estimation for  $f_{B_c}^{\text{NR}}$  is fairly in good agreement with the estimates in the framework of the lattice QCD result [8],  $f_{B_c}^{\text{NR}} = 440 \pm 20$  MeV, QCD sum rules [9], potential models [2] and also the scaling relation [15]. It indicates that the one-loop matching in Ref. [26] provides the magnitude of correction of nearly 12%. However, the recent calculation in the heavy quark potential in the static limit of QCD [13] with the one-loop matching [14] is

$$f_{B_c}^{(1\text{-loop})} = 400 \pm 15 \text{ MeV}, \quad (48)$$

Therefore, in contrast to the discussion given in [14], we see that the difference is not crucially large in our estimation to one-loop value in the  $B_c$ -meson.

Our final result for the two-loop calculations is

$$f_{B_c}^{(2\text{-loop})} = 315_{-50}^{+26} \text{ MeV}, \quad (49)$$

the larger error value in (49) is due to the strongest running coupling constant in Cornell potential.

Slightly different additive constants is permitted in this sector for a charmed mesons to bring up data to its (c.o.g.) value. However, with no additive constant to the Cornell potential [37], we notice that the smaller mass values for the composing quarks of the meson leads to a rise in the values of the potential parameters which in turn produces a notable lower value for the leptonic constant as seen in Tables II and III.

On the other hand, for the QCD motivated two types IO potential, our calculations for the ground state radial wave functions are presented in Table IV. In Table V, the NR leptonic constant of the pseudoscalar  $B_c$  state are in the range  $354 - 426$  MeV, and for the  $B_c^*$  are  $353 - 424$  MeV, for the type I. Further, they are found  $353 - 457$  and  $351 - 454$ , for the type II. For instance, we may choose  $f_{B_c}^{\text{NR}} = 391$  MeV with  $\Lambda_{\overline{MS}} = 200$  MeV, for the

type I, and  $f_{B_c}^{\text{NR}} = 396$  MeV with  $\Lambda_{\overline{MS}} = 300$  MeV, for the type II. These results surpass the matching procedure as well as methods followed in other references, cf. Refs. [8,13,14]. Further, for the CK potential, we present the decay constant in Table VI.

The scaling relation (SR) for the S-wave heavy quarkonia has the form [15]

$$\frac{f_n^2}{M_n(\bar{b}c)} \left( \frac{M_n(\bar{b}c)}{M_1(\bar{b}c)} \right)^2 \left( \frac{m_c + m_b}{4\mu} \right) = \frac{d}{n}, \quad (50)$$

where  $m_c$  and  $m_b$  are the masses of heavy quarks composing the  $B_c$ -meson,  $\mu$  is the reduced mass of quarks, and  $d$  is a constant independent of both the quark flavors and the level number  $n$ . The value of  $d$  is determined by the splitting between the 2S and 1S levels or the average kinetic energy of heavy quarks, which is independent of the quark flavors and  $n$  with the accuracy accepted. The accuracy depends on the heavy quark masses and it is discussed in [15] in detail. The parameter value in (50),  $d \simeq 55$  MeV, can be extracted from the experimentally known leptonic constants of  $\psi$  and  $\Upsilon$  [15]. Hence, in the view of Table VII, the SR gives for the 1S-level

$$f_{B_c}^{(\text{SR})} \simeq 444_{-23}^{+6} \text{ MeV}, \quad (51)$$

for all static potentials used. Kiselev estimated  $f_{B_c} = 400 \pm 45$  MeV [14] and  $f_{B_c}^{(\text{SR})} = 385 \pm 25$  MeV [15], Narison found  $f_{B_c}^{(\text{SR})} = 400 \pm 25$  MeV [38].

Overmore, we give the leptonic constants for the excited nS-levels of the  $\bar{b}c$  quarkonium system in Table VII. We remark that the calculated value of  $f_{B_c(2S)}^{(\text{SR})} = 300 \pm 15$  MeV is in good agreement with  $f_{B_c(2S)}^{(\text{SR})} = 280 \pm 50$  MeV being calculate in [13] for the 2S-level in the  $\bar{b}c$  system and it is also consistent with the scaling relation [15].

In Figure 1, we plot the calculated values of  $B_c$  leptonic constants using (50) for different potential models together with the calculated values of the excited nS-states using [15]

$$f_n = \frac{1}{\sqrt{n}} f_1. \quad (52)$$

Hence, our findings obtained from the potential model and SR are in good agreement to several MeV as shown in Figure 1.

Finally, we have also noted that the leptonic constant is practically independent of the total spin of quarks, so that

$$f_{V,n} \simeq f_{P,n} = f_n, \quad (53)$$

where  $M_{B_c} \simeq m_b + m_c$ . Thus, one can conclude that for the heavy quarkonium, the QCD sum rule approximation gives the identity of leptonic constant values for the pseudoscalar and vector states.

In this work, we have successfully applied the SLNET using a class of static and QCD-motivated potentials to calculate numerically the leptonic constant of the pseudoscalar and vector  $B_c$ -meson. Once the experimental leptonic constant of the  $B_c$  meson becomes clear, one can sharpen the analysis.

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## TABLES

TABLE I. The characteristics of the wave function  $|\psi_{1S}(0)|^2$  and the radial wave function  $|R_{1S}(0)|^2 = 4\pi |\psi_{1S}(0)|^2$  both at the origin (in  $\text{GeV}^3$ ) obtained from the Schrödinger equation for various potentials.

Level	Cornell	Song-Lin	Turin	Martin	Logarithmic	Cornell <sup>a</sup>
$ \psi_{1S}(0) ^2$	0.112	0.123	0.111	0.119	0.102	0.065
$ R_{1S}(0) ^2$	1.413	1.54	1.397	1.495	1.28	0.814

<sup>a</sup>Here we cite Ref. [37] for the fitted set of parameters.

TABLE II. Pseudoscalar and vector decay constants ( $f_P = f_{B_c}$ ,  $f_V = f_{B_c^*}$ ) of the  $B_c$  meson, calculated in the different potential models (the accuracy is 5%), in MeV.

Constants	Cornell <sup>a</sup>	Song-Lin	Turin	Martin	Logarithmic	Cornell <sup>b</sup>	[35]	[36]
$f_{B_c}^{NR}$	464.5	485.1	462.0	478.1	441.7	351.5	$460 \pm 60$	$420 \pm 13$
$f_{B_c^*}^{NR}$	461.5	482.2	459.2	475.3	439.3	349.7	$460 \pm 60$	

<sup>a</sup>Here  $V_0 \neq 0$ .

<sup>b</sup>Here  $V_0 = 0$ , see Ref. [37].

TABLE III. One-loop and two-loop corrections to pseudoscalar and vector decay constants of the low-lying  $\bar{b}c$  system, calculated in the different potential models (the accuracy is 4%), in MeV.

Quantity	Cornell	Song-Lin	Turin	Martin	Logarithmic	Cornell
$f_{B_c}^{(1-loop)}$	393.6 <sup>a</sup>	424.4	399.6	421.2	399.3	311.9
$f_{B_c}^{(2-loop)}$	264.1 <sup>b</sup>	333.0	296.6	339.2	340.9	238.1
$f_{B_c^*}^{(1-loop)}$	391.0	421.9	397.1	418.7	397.1	310.3
$f_{B_c^*}^{(2-loop)}$	262.4	331.0	294.8	337.2	339.0	236.9

<sup>a</sup>To the first correction for all potentials,  $K_0 = 0.85 - 0.90$ .

<sup>b</sup>To the second correction for all potentials,  $K_0 = 0.57 - 0.77$ .

TABLE IV. The characteristics of the radial wave function at the origin  $|R_{1S}(0)|^2$  (in  $\text{GeV}^3$ ) obtained from the Schrödinger equation for the Igi-Ono potential model.

Level <sup>a</sup>	$\Lambda_{\overline{MS}}$				
	100	200	300	400	500
Type I <sup>b</sup>					
$ \psi_{1S}(0) ^2$	0.066	0.08	0.092	0.095	0.089
$ R_{1S}(0) ^2$	0.826	1.005	1.156	1.19	1.114
Type II <sup>c</sup>					
$ \psi_{1S}(0) ^2$	0.065	0.071	0.082	0.096	0.109
$ R_{1S}(0) ^2$	0.819	0.891	1.03	1.204	1.366

<sup>a</sup>For the fitted parameters, see Ref. [17].

<sup>b</sup>We shifted the  $\bar{c}c$  spectra by  $c = -22$  to  $-31 \text{ MeV}$ .

<sup>c</sup>We shifted the  $\bar{c}c$  spectra by  $c = -15$  to  $-26 \text{ MeV}$ .

TABLE V. Pseudoscalar and vector decay constants of the  $B_c$  meson, calculated in the Igi-Ono potential model, in MeV.

Quantity <sup>a</sup>	$\Lambda_{\overline{MS}}$				
	100	200	300	400	500
Type I <sup>b</sup> : $\alpha_s =$	0.199 <sup>c</sup>	0.217	0.238	0.25	0.262
$f_{B_c}^{NR}$	354.1	391.1	420.0	426.0	411.7
$f_{B_c^*}^{NR}$	352.6	389.2	417.7	423.6	409.4
Type II <sup>d</sup> : $\alpha_s =$	0.199	0.227	0.230	0.241	0.251
$f_{B_c}^{NR}$	352.7	368.2	396.1	428.6	456.9
$f_{B_c^*}^{NR}$	351.2	366.4	394.1	426.4	454.4

<sup>a</sup>For the fitted parameters we cite Ref. [17].

<sup>b</sup>We shifted the  $\bar{c}c$  spectrum by  $c = -22$  to  $-31$  MeV.

<sup>c</sup>Input value  $\Delta E_{HF} = 117$  MeV; determines  $\alpha_s(m_c)$ .

<sup>d</sup>We shifted the  $\bar{c}c$  spectra by  $c = -15$  to  $-26$  MeV.

TABLE VI. The radial wave function, pseudoscalar and vector decay constants of the  $B_c$  meson, calculated in the improved Chen-Kuang potential model, in MeV.

Quantity <sup>a</sup>	100	180	300	350	375	400	450	500
$ R_{1S}(0) ^2$	1.017	1.017	1.017	1.017	1.017	1.015	0.957	0.829
$f_{B_c}^{NR}$	393.5	393.5	393.5	393.4	392.5	389.9	381.8	353.8
$f_{B_c^*}^{NR}$	391.4	391.4	391.4	391.3	390.4	387.8	379.8	352.3

<sup>a</sup>Input value  $\Delta E_{HF} = 117$  MeV; determines  $\alpha_s(m_c) = 0.27$ .

TABLE VII. The nS-levels leptonic constant of the  $\bar{b}c$  system, calculated in the different potential models (the accuracy is 3 – 7%), in MeV, using the SR.

Quantity	Cornell	Song-Lin	Turin	Martin	Logarithmic
$f_{1S}$	449.6	450.4	448.0	448.8	420.9
$f_{2S}$	305.8	305.0	303.3	303.5	284.7
$f_{3S}$	243.0	243.2	241.3	241.8	227.2
$f_{4S}$	206.0	207.1	204.9	205.9	193.8

## FIGURES

FIG. 1. The  $nS$ -levels leptonic constant of the  $B_c$  system calculated in the different potential models using SR.